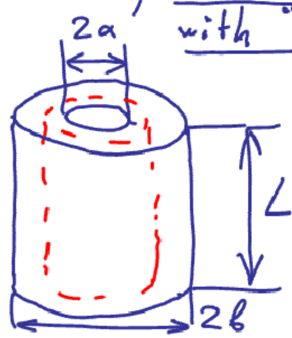
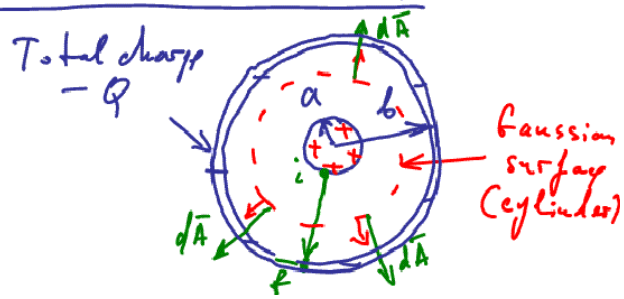
Problem 1

Find the expression for the capacitance of an air-filled cylindrical capacitor with inner-shell radius "a" and outer-shell radius "b"

a) Sketch of the geometry of the problem with "a" and "b" indicated





b) Full sequence of steps

Assume total charge Q: +Q on inner-shell
−Q on outer-shell

$$Q \xrightarrow{\text{Gauss's Law}} E(r) \xrightarrow[\text{of } V]{\text{Use the definition}} \Delta V \xrightarrow[C = \frac{Q}{\Delta V}]{\text{Use the definition of } C} C$$

c) Finding E(r)

$$\Phi_e = \frac{Q_{enc}}{\varepsilon_0}$$

$$\Phi_e = \oint_{\text{Gaussian surface}} \bar{E} \cdot d\bar{A} = \oint_G E(r) \cdot dA = E(r) \oint_G dA =$$

$$\boxed{\bar{E} \cdot d\bar{A} = |\bar{E}| \cdot |d\bar{A}| \cdot \cos\varphi = |\bar{E}| \cdot |d\bar{A}| = E(r) \cdot dA \quad \varphi = 0^\circ \Rightarrow \cos\varphi = 1}$$

$E(r) \cdot 2\pi \cdot r \cdot L.$

$Q_{enc} = Q$ – assumed charge

$$E(r) \cdot 2\pi r \cdot L = \frac{Q}{\varepsilon_0} \Rightarrow E(r) = \frac{1}{2\pi\varepsilon_0} \frac{Q}{r \cdot L}.$$

d) Calculating ΔV

$$V_f - V_i = -\int_i^f \bar{E} \cdot d\bar{s} \quad \text{– definition of } V$$

$$V_f - V_i = -\int_a^b \frac{1}{2\pi\varepsilon_0} \frac{Q}{r \cdot L} dr =$$

$\boxed{\text{We switch from } ds \text{ to } dr, \ dr = ds}$

$$= -\frac{Q}{2\pi\varepsilon_0 \cdot L} \int_a^b \frac{dr}{r} = -\frac{Q}{2\pi\varepsilon_0 \cdot L} \ln r \Big|_a^b =$$

$$= \frac{-Q}{2\pi\varepsilon_0 \cdot L} (\ln b - \ln a) = -\frac{Q}{2\pi\varepsilon_0 \cdot L} \ln \frac{b}{a}$$

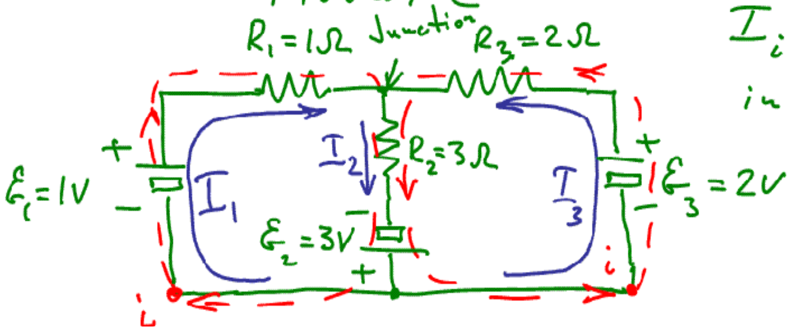
$V_f - V_i < 0$

$$\Delta V = V_i - V_f, \quad \Delta V = \frac{Q}{2\pi\varepsilon_0 \cdot L} \ln \frac{b}{a}$$

e) Calculating C:

$$C = \frac{Q}{\Delta V} = \frac{\cancel{Q} \cdot 2\pi\varepsilon_0 \cdot L}{\cancel{Q} \cdot \ln \frac{b}{a}} = \frac{2\pi\varepsilon_0 \cdot L}{\ln \frac{b}{a}}$$

Problem 2



Find currents I_i $i = 1, 2, 3$ in all branches

a) Sketch of the assumed directions of currents in all branches

b) Sketch of the directions of moving along the loops: clockwise or counterclockwise.

c) Equations required for solving problem for assumed directions:

$$2 - I_3 \cdot 2 - I_2 \cdot 3 + 3 = 0$$ − Loop Rule (right loop)

$$5 = 2I_3 + 3I_2 \quad (1)$$

$$1 - I_1 \cdot 1 - I_2 \cdot 3 + 3 = 0$$ − Loop Rule (left loop)

$$4 = I_1 + 3I_2 \quad (2)$$

$$I_1 + I_3 = I_2 \quad (3)$$ − Junction Rule

(3) in (2): $4 = I_1 + 3(I_1 + I_3) = 4I_1 + 3I_3 \quad (4)$

(3) in (1): $5 = 2I_3 + 3(I_1 + I_3) = 3I_1 + 5I_3 \quad (5)$

$\downarrow$ from (5)

$$I_1 = \frac{5 - 5I_3}{3} = \frac{5}{3}(1 - I_3) \quad (5^*)$$

Subst. in (4)

$$4 = \frac{4 \cdot 5}{3}(1 - I_3) + 3I_3$$

$$4 = \frac{20}{3} - \frac{20}{3} I_3 + 3I_3 = \frac{20}{3} - \frac{20-9}{3} I_3 =$$

$$= \frac{20}{3} - \frac{11}{3} I_3$$

$$I_3 = \left(\frac{20}{3} - 4\right)\frac{3}{11} = \frac{20-12}{\cancel{3}} \cdot \frac{\cancel{3}}{11} = \frac{8}{11}\,(A)$$

From (5^*)

$$I_1 = \frac{5}{3}(1 - I_3) = \frac{5}{3}\left(1 - \frac{8}{11}\right) = \frac{5}{3} \cdot \frac{11-8}{11} =$$

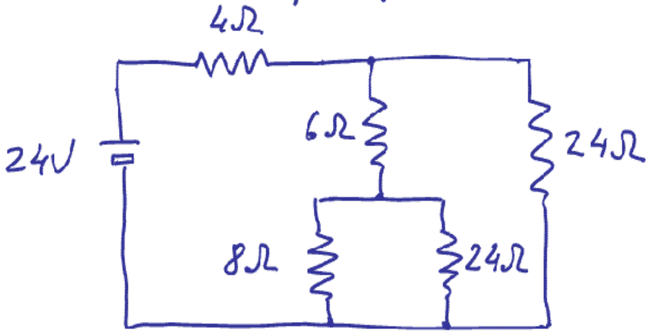
$$= \frac{5}{\cancel{3}} \cdot \frac{\cancel{3}}{11} = \frac{5}{11}\,(A)$$

From (3):

$$I_2 = I_1 + I_3 = \frac{8}{11} + \frac{5}{11} = \frac{13}{11}\,A$$

Problem 3

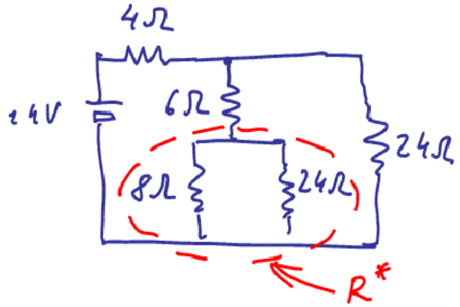
Sample problem 31-69



Find I_i and V_i for each resistor. Put your answers in a Table.

a) Sketch all transformations of the circuit based on the laws of series and parallel resistors

b) Provide solution for each step



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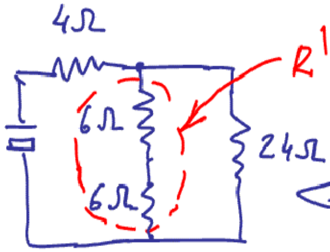

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$$\frac{1}{R^*} = \frac{1}{8} + \frac{1}{24} = \frac{3+1}{24} = \frac{4}{24} = \frac{1}{6} \Rightarrow R^* = 6\,\Omega$$

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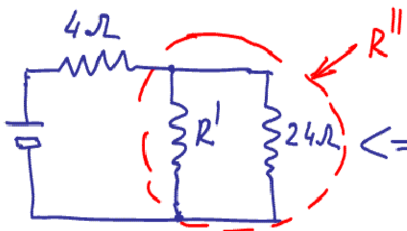

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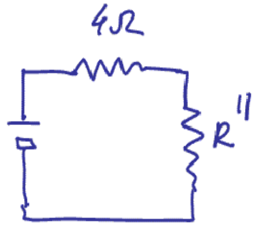

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$$R' = 6\,\Omega + 6\,\Omega = 12\,\Omega$$

$$\frac{1}{R''} = \frac{1}{12} + \frac{1}{24} = \frac{2+1}{24} = \frac{3}{24} = 1/8 \Rightarrow R'' = 8\,\Omega$$

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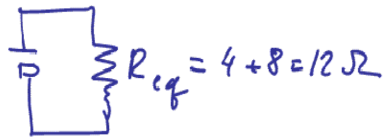
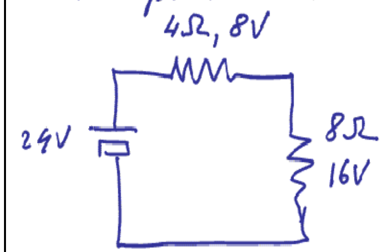


We gained the value of total current

$$I = \frac{24V}{12\,\Omega} = 2\,A$$

As we rebuild the circuit, we note that series resistors must have the same current and the parallel resistors must have the same ΔV.



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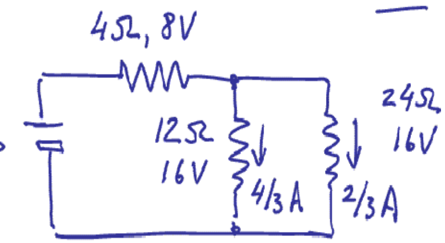


$\Delta V = I \cdot R$

$I = \frac{\Delta V}{R}$

Step 1

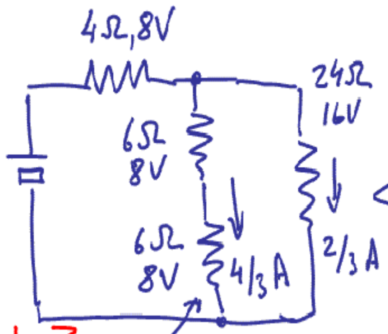
The 12 Ω resistor is returned to 4Ω and 8Ω resistors in series. Both resistors must have the same 2.0 A current as the 12 Ω resistor

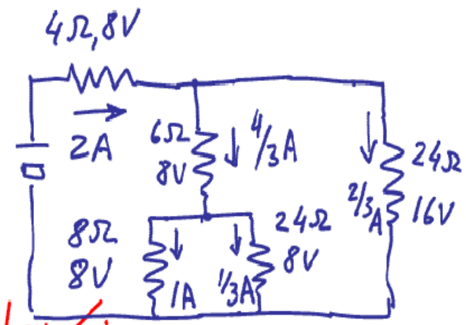
Step 2

The 8Ω resistor is returned to the 12 Ω and 24 Ω resistors in parallel. Both resistors must have the same $\Delta V = 16V$ as the 8Ω resistor. From Ohm's law,

$$I_{12} = (16V)/(12\,\Omega) = \frac{4}{3}A$$

$$I_{24} = \frac{2}{3}\,A$$





Step 3

12Ω resistor is returned to the two 6Ω resistors in series. Both resistors must have the same $\frac{4}{3}A$ as the 12Ω resistor. Due to Ohm's law their $\Delta V_6 = I \cdot R = 8V$ for each resistor

Step 4

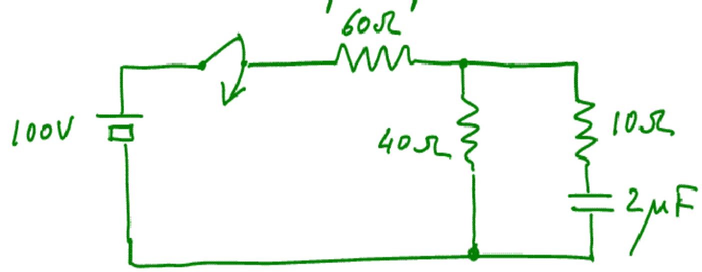
The 6Ω resistor is returned to the 8Ω and 24Ω resistors in parallel. Both resistors must have the same $\Delta V = 8V$ as the 6Ω resistor. From Ohm's law $I_8 = (8V)/(8\Omega) = 1A$ and $I_{24} = 1/3\ A$.

Table of answers

Resistor	Potential Difference ΔV (V)	Current (A)
4Ω	8	2
6Ω	8	4/3
8Ω	8	1
Bottom 24Ω	8	1/3
Right 24Ω	16	2/3

Problem 4

Sample problem 31-77



The switch has been closed for a long time

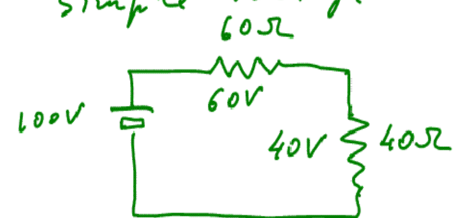
a) What is the charge Q_c - ?

b) The switch is opened at $t = 0s$

At what time has the charge on the capacitor decreased to 10% of its initial value?

__Understanding of the steady state of the circuit with the switch being closed for a long time.__

The capacitor is fully charged and there is no current in the right branch that contains C. So, the capacitor can be modeled as a break, and the circuit becomes to be a simple voltage divider:



$$I = \frac{100V}{60\Omega + 40\Omega} = 1A$$

$$\Delta V = I \cdot R$$

So, $\Delta V = 40V$ appears across the 40Ω resistor.

This voltage is applied to the capacitor.
So, the charge on the capacitor is

$$Q_0 = (\Delta V) \cdot C = (40V) \cdot (2 \cdot 10^{-6} F) = 80 \mu C$$

Mathematical description of the transient process occuring after opening the switch

Once the switch is opened, the battery is disconnected.
The capacitor C has two resistors (10Ω and 40Ω) in series and discharges according to

$$Q_C = Q_0 \cdot e^{-\frac{t}{\tau}}, \quad \text{where } \tau = RC$$
$$R = 10\Omega + 40\Omega = 50\Omega$$

$$\tau = 50\Omega \cdot 2 \cdot 10^{-6} F = 100 \cdot 10^{-6} s = 0.1 ms$$

For $Q = 0.1 \cdot Q_0$

$$0.1 \cdot Q_0 = Q_0 \cdot e^{-\frac{t}{\tau}} \Rightarrow 0.1 = e^{-\frac{t}{\tau}}$$

$$\ln 0.1 = -\frac{t}{\tau}$$

$$-2.3 = -\frac{t}{\tau} \Rightarrow t = 2.3 \cdot \tau = \underline{\underline{0.23 ms}}$$